Total number of printed pages–11

3 (Sem–1/CBCS) MAT HC 2

2020

(Held in 2021)

MATHEMATICS

(Honours)

Paper : MAT–HC–1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (a) Find the polar representation of the point $(2,-2)$.

   (b) Find the Cartesian co-ordinates of the point $\left(2, {}^{2\pi}\!/_{3}\right)$.

   (c) If $f : R \to R$ is given by $f(x) = x^2$, what is $f^{-1}((0, 4))$?

Contd.

(d) Write the statement and its negation using quantifiers.
"In each tree in the garden, we can find a branch on which all of the leaves are green".

(e) If A is the set of all $n \times n$ symmetric matrices and B is the set of all $n \times n$ real skew-symmetric matrices, what is $A \cap B$?

(f) Let $M(2, \mathbb{R})$ denote the set of all 2×2 matrices over $\mathbb{R}$.
Consider the function $f : M(2, \mathbb{R}) \rightarrow \mathbb{R}$ given by $f(A) = det\, A$.
Show that f is not one-one.

(g) State the well-ordering principle in $\mathbb{N}$.

(h) *State 'true'* ***or*** *'false' with justification :*
If one row in an echelon form of an augmented matrix is $[0\ 0\ 0\ 5\ 0]$, then the associated linear system is inconsistent.

(i) *State 'true'* ***or*** *'false' with justification :*
Each column of AB (where A and B are matrices whose product AB is defined) is a linear combination of the columns of B using weights from the corresponding columns of A.

(j) *Fill in the blanks :*
If A is a triangular matrix then *det A* is the product of the entries on the ________ ________.

2. Answer the following questions : $2\times5=10$

(a) Compute $(1+i)^{100}$.

(b) Describe the following set explicitly and mark it on the real line

$$X=\{x\in\mathbb{R} \mid x(x-1)(x-2)<0\}$$

(c) Consider the relation on $\mathbb{R}$ defined by $R=\{(x,y)\in\mathbb{R}\times\mathbb{R} \mid x\leq y\}$. Is this relation an equivalence relation? Justify.

(d) Find standard matrix of T, where T is a linear transformation such that $T:\mathbb{R}^2\to\mathbb{R}^2$ rotates points about the origin through $3\pi/2$ in a counter-clockwise manner.

(e) A is 3×3 matrix with three pivot positions. Explain the following—

(i) Does $A\vec{x}=\vec{0}$ have a nontrivial solution?

(ii) Does $A\vec{x}=\vec{b}$ have at least one solution for all $\vec{b}$ in $\mathbb{R}^3$?

3. Answer ***any four*** questions : 5×4=20

(a) If $n|q$ then prove that any root of $z^n - 1 = 0$ is a root of $z^q - 1 = 0$. Prove that the common roots of $z^m - 1 = 0$ and $z^n - 1 = 0$ are roots of $z^d - 1 = 0$ where $d =$ g.c.d (m, n)

i.e. $U_m \cap U_n = U_d$. 1+4=5

(b) Let $X = \mathbb{R} = Y$. Let $A = \{1\}$ and $B = \mathbb{R}$. Draw the sketch of $A \times B$ as a subset of $\mathbb{R}^2$.

For $A \subseteq X$ and $B \subseteq Y$ show that there may be subsets of $X \times Y$ that are not of the form $A \times B$. 2+3=5

(c) For any sets *A* and *B*, show that the following are equivalent. 5

(i) $A \subseteq B$

(ii) $A \cup B = B$

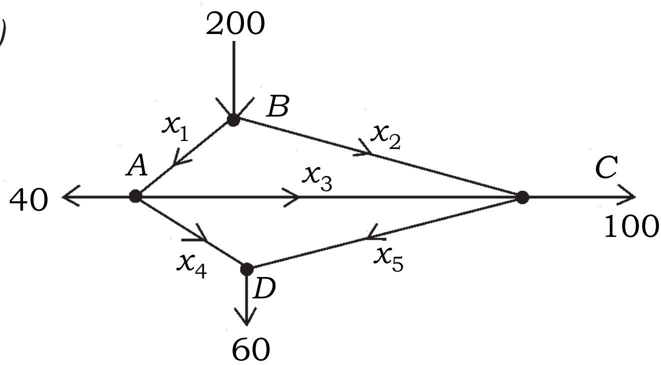
(iii) $A \cap B = A$

(iv) $B^c \subseteq A^c$

(d) Describe the solutions of the following system in parametric form. 5

$$
\begin{aligned}
x_1 + 3x_2 - 5x_3 &= 4 \\
x_1 + 4x_2 - 8x_3 &= 7 \\
-3x_1 - 7x_2 + 9x_3 &= -6
\end{aligned}
$$

(e)



For the figure above find the general traffic pattern in the freeway network (Flow rates are cars/minute)

Describe the general traffic pattern when the road whose flow is x_4 is closed. 5

(f) Use Cramer's Rule to compute the solutions of the system 5

$$
\begin{aligned}
2x_1 + x_2 + x_3 &= 4 \\
-x_1 + \quad + 2x_3 &= 2 \\
3x_1 + x_2 + 3x_3 &= -2
\end{aligned}
$$

4. Answer ***any four*** of the following : 10×4=40

(a) (i) Compute $z^n + \frac{1}{z^n}$ if $z + \frac{1}{z} = \sqrt{3}$. 5

(ii) Prove that 5

$$sin\,5t = 16\,sin^5\,t - 20\,sin^3\,t + 5\,sin\,t$$
$$cos\,5t = 16\,cos^5\,t - 20\,cos^3\,t + 5\,cos\,t$$

(b) (i) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (ax + by, cx + dy)$ is a bijection of $ad - bc \neq 0$. Find the inverse of *f*. 5

(ii) For any sets *A*, *B*, *C* prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 5

(c) (i) Show that a function $f : X - Y$ is one-one if and only if $f(A \cap B) = f(A) \cap f(B)$ holds for all subsets *A* and *B* of *X*. 6

(ii) Give an example of a relation that is reflexive and transitive but not symmetric. 2

(iii) Produce a counter example to disapprove the statement —
"For integers a, b, c if a divides bc, then a divides b or a divides c." 2

(d) Prove that if a, b are integers not both zero and d be the greatest common divisor of a and b then $\exists$ integers x, y s.t. $d = ax + by$. Further, prove that two integers m and n are relatively prime if and only if $\exists$ integers p and q s.t. $pm + qn = 1$. 6+4

(e) (i) Prove the statement using contrapositive
"For integers x, y if $x + y$ is even, then x and y are both odd or both even". 2

(ii) Determine h such that the following matrix is the augmented matrix of a consistent linear system

$$\begin{bmatrix} 2 & 8 & h \\ 4 & 6 & 7 \end{bmatrix}$$ 2

(iii) Determine if $\vec{b}$ is a linear combination of the vectors formed from the column of A. 4

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

(iv) Show that the set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. 2

(f) (i) If A is a $m \times n$ matrix and $\vec{U}, \vec{V}$ are vectors in $\mathbb{R}^n$, C is a scalar then prove that

$$A\left(\vec{U} + \vec{V}\right) = A\vec{U} + A\vec{V}$$

and $A\left(C\vec{U}\right) = C\left(A\vec{U}\right)$ 4

(ii) If A is a square $n \times n$ matrix then prove that the following statements are logically equivalent :

(a) A is an invertible matrix

(b) There is an $n \times n$ matrix C such that $CA = I$.

(c) The equation $A\vec{x} = \vec{0}$ has only the trivial solution.

(d) A has n pivot positions

(e) A is row equivalent to the $n \times n$ identity matrix. 6

(g) *(i)* Write the following system in matrix form and use the inverse of the co-efficient matrix to solve

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$ 2

(ii) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T.
Then show that T is invertible if and only if A is an invertible matrix. Show that the linear transformation S given by $S(\vec{x}) = A^{-1}\vec{x}$ where $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$, is the unique inverse of T. 4

(iii) Find the inverse of the following matrix if it exists by performing suitable row operations on the augmented matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$ 4

(h) *(i)* Compute the determinant by co-factor expansion choosing the row or column that involves least amount of computation

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$$ 2

(ii) *State 'true'* ***or*** *'false' with justification :*

The $(i, j)^{th}$ cofactor of a matrix A is the matrix A_{ij} obtained from A by deleting the i^{th} row and j^{th} column of A. 2

(iii) For the matrix given below

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

(a) compute the determinant.

(b) what is determinant of an elementary row replacement of the matrix ?

(c) what is the determinant of an elementary scaling matrix with k on the diagonal ? 3

(iv) Use a determinant to decide if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, where

$$\vec{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

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