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3 (Sem-5/CBCS) MAT HE1/2/3

2024

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION - A

Paper : MAT-HE-5016

(Number Theory)

OPTION - B

Paper : MAT-HE-5026

(Mechanics)

OPTION - C

Paper : MAT-HE-5036

(Probability and Statistics)

Full Marks : 80

Time : Three hours

***The figures in the margin indicate
full marks for the questions.***

Contd.

OPTION - A

Paper : MAT-HE-5016

(Number Theory)

1. Answer the following questions as directed :
 $1 \times 10 = 10$

(a) State Goldbach conjecture.

(b) If p and q are twin primes, then which of the following statements is true ?

(i) $pq = (p+1)^2 - 1$

(ii) $pq = (p+1)^2 + 1$

(iii) $pq = (p-1)^2 + 1$

(iv) None of the above

(c) Give an example to show that $a^2 \equiv b^2 \pmod{n}$ need not imply that $a \equiv b \pmod{n}$.

(d) State whether the following statement is True **or** False :

“The polynomial function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = n^2 + n + 41$ provides only prime numbers.”

- (e) Define a pseudoprime number.
- (f) Find the sum of all positive divisors of 360.
- (g) Which of the following is a perfect number ?

(i) 9

(ii) 10

(iii) 18

(iv) 28

(h) If x is not an integer then find the value of $[x] + [-x]$.

(i) State whether the following statement is True **or** False :

“If $\tau(n)$ is an odd integer, then $\sqrt{(n)^{\tau(n)}}$ is not an integer.”

(j) Find the number of integers less than 900 and prime to 900.

2. Answer the following questions : $2 \times 5 = 10$

(a) Find the remainder when 41^{65} is divided by 7.

(b) If $a \equiv b \pmod{n}$, then show that $a - m \equiv b - m \pmod{n}$, where m is any integer.

(c) Show that any prime of the form $3k+1$ is also of the form $6k+1$, where k is an integer.

(d) If n is an odd positive integer, then prove that $\phi(2n) = \phi(n)$.

(e) For $n \geq 3$, evaluate $\sum_{k=1}^n \mu(n!)$

3. Answer **any four** questions : 5×4=20

(a) Show that $a \equiv b \pmod{n}$ if and only if a and b have same remainder on division by n .

(b) Show that there are infinite number of primes of the form $4n+3$.

(c) Solve using Chinese Remainder Theorem the simultaneous congruences :

$$x \equiv -2 \pmod{12}; x \equiv 6 \pmod{10}; x \equiv 1 \pmod{15}$$

- (d) If p is a prime and n is a positive integer, then show that the exponent e such that

$$p^e / n! \text{ is at most } \sum_{i=1}^{\infty} \left[\frac{n}{p^i} \right].$$

- (e) Show that the system of linear congruences

$$ax + by \equiv r \pmod{n}$$

$$cx + dy \equiv s \pmod{n}$$

has a unique solution modulo n whenever $\gcd(ad - bc, n) = 1$.

- (f) If $n \geq 1$ is an integer, then show that $\sigma(n)$ is odd $\Leftrightarrow n$ is a perfect square or twice a perfect square.

4. (a) State and prove Fermat's theorem. Is the converse of this theorem true? Justify your answer. 1+5+4=10

OR

- (b) (i) Show that every integer $n > 1$, is either a perfect square or the product of a square-free integer and a perfect square. 5

(ii) Let

$$n = a_m (1000)^m + a_{m-1} (1000)^{m-1} + \dots + a_1 (1000) + a_0$$

where a_k 's are integers such that

$$0 \leq a_k \leq 999 \text{ and } T = \sum_{k=0}^m (-1)^k a_k.$$

Prove that n is divisible by 7 if and only if T is divisible by 7. 5

5. (a) (i) If p is a prime then show that $(p-1)! \equiv -1 \pmod{p}$. Also verify it for $p = 13$. 4+3=7

- (ii) Show that any integer of the form $8^n + 1$ is not a prime. 3

OR

- (b) State and prove Fundamental Theorem of Arithmetic. Also find a prime number p such that $2p+1$ and $4p+1$ are also primes. 1+6+3=10

6. (a) (i) For each positive integer $n \geq 1$, prove that

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

7

- (ii) If n is the product of a pair of twin primes, prove that

$$\phi(n)\sigma(n) = (n+1)(n-3). \quad 3$$

OR

- (b) (i) If f is a multiplicative arithmetic function, then show that

$$g_1(n) = \sum_{d|n} f(d) \quad \text{and}$$

$$g_2(n) = \sum_{d|n} \mu(d) f(d)$$

are both multiplicative arithmetic functions. 7

- (ii) If n is an even positive integer, then prove that $\phi(2n) = 2\phi(n)$. 3

7. (a) (i) Define Möbius pair. If (f, g) is a Möbius pair and either f or g is multiplicative then show that both f and g are multiplicative. 2+3=5

- (ii) If p is a prime number and k is any positive integer, then show that

$$\phi(p^k) = p^k \left(1 - \frac{1}{p}\right). \quad 5$$

OR

(b) (i) If x and y are real number then show that

$$[x] + [y] \leq [x + y] \leq [x] + [y] + 1. \quad 5$$

(ii) If $n = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots \cdot p_r^{m_r}$ where p_i 's are distinct primes and $m_i \in N, m_i \geq 1$ then for each $r \geq 1$ prove that

$$\tau(n) = \prod_{i=1}^r (m_i + 1). \quad 5$$

OPTION - B

Paper : MAT-HE-5026

(*Mechanics*)

1. Answer the following questions: $1 \times 10 = 10$

- (a) State the parallelogram law of forces.
- (b) Can a force and a couple acting in one plane maintain equilibrium ?
- (c) What is the position of the centre of gravity of a uniform triangular lamina ?
- (d) Write down the relationship between the co-efficient of friction and the angle of friction.
- (e) What is the physical significance of moment of a force about a point ?
- (f) Write the expressions for radial and transverse components of acceleration of a particle moving in a plane curve.
- (g) Define a conservative force field. Give one example.
- (h) State the principle of conservation of energy.

Contd.

- (i) A particle moves in a straight line from a distance a towards the centre of force the force being varies inversely as the cube of the distance. Write down the equation of motion.
- (j) State Hooke's law of elasticity.

2. Answer the following questions : $2 \times 5 = 10$

- (a) Find the angle between two equal forces each equal to P , when their resultant is a third equal force P .
- (b) Two men are carrying a straight uniform bar 6 m long and weighing 30 kg . One man supports it at a distance of 1 m from one end and the other man at a distance of 2 m from the other end. What weight does each man bear ?
- (c) Prove that the centre of gravity of a body is unique.
- (d) An impulse I changes the velocity of a particle of mass m from v_1 to v_2 . Show that the kinetic energy gained is $\frac{1}{2}I \cdot (v_1 + v_2)$.

(e) State Newton's 2nd law of motion. How does the 2nd law of motion give us a method to measure force ?

3. Answer the following questions : **(any four)**
 $5 \times 4 = 20$

(a) Force $\vec{P}$, $\vec{Q}$, $\vec{R}$ acting along $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, where O is the circum-centre of the triangle ABC , are in equilibrium. Show that

$$\frac{P}{a^2(b^2 + c^2 - a^2)} = \frac{Q}{b^2(c^2 + a^2 - b^2)} = \frac{R}{c^2(a^2 + b^2 - c^2)}$$

(b) Prove that, any system of coplanar forces acting on a rigid body can be reduced ultimately to either a single force or a single couple unless it is in equilibrium.

(c) A uniform ladder rests in limiting equilibrium with the lower end on a rough horizontal plane and its upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, then prove that $\tan \theta = 2\mu$, where μ is the co-efficient of friction.

Contd.

(d) A particle is constrained to move along the equiangular spiral $r = ae^{b\theta}$ so that the radius vector moves with constant angular velocity ω . Determine the velocity and acceleration components.

(e) A particle of mass m is acted upon by a

force $m\mu \left(x + \frac{a^4}{x^3} \right)$ towards the origin. If

it starts from rest at a distance a , show that it will arrive at the origin in time

$$\frac{\pi}{4\sqrt{\mu}}.$$

(f) A particle falls under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity. If the particle starts from rest, derive the expression for velocity of the particle at the end of time t .

4. Answer the following questions : **(any four)**

$$10 \times 4 = 40$$

(a) (i) If the resultant of two equal forces inclined at an angle 2θ is twice as great as when they are inclined at an angle 2ϕ , then prove that $\cos \theta = 2 \cos \phi$.

5

- (ii) P and Q are two like parallel forces. If a couple, each of whose forces is F and whose arm is a in the plane of P and Q , is combined with them, then show that the resultant is displaced through a distance

$$\frac{Fa}{P+Q} \quad 5$$

- (b) (i) Prove that, if three coplanar forces acting on a rigid body be in equilibrium, then they must either all three meet at point, or else all must be parallel to one another.

4

- (ii) Force P, Q, R act along the sides $\overline{BC}, \overline{CA}, \overline{AB}$ of the triangle ABC and forces P', Q', R' act along $\overline{OA}, \overline{OB}, \overline{OC}$, where O is the circum-centre, in the senses indicated by the order of the letters. If the six forces are in equilibrium, then show that $P \cos A + Q \cos B + R \cos C = 0$

$$\text{and } \frac{PP'}{a} + \frac{QQ'}{b} + \frac{RR'}{c} = 0 \quad 6$$

(c) (i) Define force of friction. What is limiting friction? State the laws of statical friction and limiting friction.

1+1+3=5

(ii) A body of weight W rests on a rough horizontal plane, λ being the corresponding angle of friction. It is desired to move the body on the plane by pulling it with the help of a string. Find the least angle of friction and the least force necessary.

5

(d) (i) Find the centre of gravity of the arc of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

lying in the first quadrant.

5

(ii) Find the centre of gravity of the solid formed by revolving $r = a(1 + \cos \theta)$ about the x -axis.

5

- (e) A particle P , of mass m , moves in a straight line OX under a force $m\mu$ (distance) directed towards a point A which moves in the straight line OX with constant acceleration a . Show that the motion of P is simple harmonic of

period $\frac{2\pi}{\sqrt{\mu}}$, about a moving centre which

is always at a distance $\frac{a}{\mu}$ behind A .

10

- (f) One end of an elastic string, whose modulus of elasticity is λ and whose unstretched length is a , is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass m which is lying on the table. The particle is pulled to a distance where the extension of the string is b and then let go; show that the time of a complete oscillation is

$$2\left(\pi + \frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}$$

(g) (i) Show that the path of a point P which possesses two constant velocities u and v , the first of which is in a fixed direction and the second of which is perpendicular to the radius OP drawn from a fixed point O , is a conic whose focus is O and whose eccentricity is $\frac{u}{v}$. 5

(ii) A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent; show that the curve is an equiangular spiral. 5

(h) A particle falls from rest under gravity through a distance x in a medium whose resistance varies as the square of the velocity. If v is the velocity actually acquired by it, v_0 is the velocity it would have acquired had there been no resistance and V is the terminal velocity, show that

$$\frac{v^2}{V_0^2} = 1 - \frac{1}{2} \frac{v_0^2}{V^2} + \frac{1}{2.3} \frac{v_0^4}{V^4} - \frac{1}{2.3.4} \frac{v_0^6}{V^6} + \dots$$

OPTION - C

Paper : MAT-HE-5036

(Probability and Statistics)

1. Answer the following questions as directed :
 $1 \times 10 = 10$

(a) Find the total number of elementary events associated to the random experiment of throwing three dice.

(b) Define probability density function for a continuous random variable.

(c) If $P(x) = 0.1x$, $x = 1$
 0 , otherwise

find $P\{x = 1 \text{ or } x = 2\}$.

(d) If X and Y are two random variables and $\text{var}(X - Y) \neq \text{var}(X) - \text{var}(Y)$ then what is the relation between X and Y ?

(e) Can the probabilities of three mutually exclusive events A , B , C as given by

$$P(A) = \frac{2}{3}, P(B) = \frac{1}{4} \text{ and } P(C) = \frac{1}{6} \text{ be}$$

correct? If not, give reason.

(f) Mention the relationship among the mean, median and mode of the normal distribution.

- (g) Under what condition $cov(X, Y) = 0$?
- (h) If X and Y are two independent random variables, then find $var(2x + 3y)$.
- (i) Write the equation of line of regression of x on y .
- (j) If a non-negative real-valued function f , which is the probability density function of the continuous random variable X , is given by $f(x) = 2x, 0 \leq x \leq 1$ and $P(x \geq a) = P(x > a)$, then find a .

2. Answer the following questions : $2 \times 5 = 10$

- (a) If A and B are independent events then show that A and $\bar{B}$ are also independent.
- (b) Find k , such that the function f defined by

$$f(x) = \begin{cases} kx^2 & \text{when } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function. Also

determine $P\left(\frac{1}{3} < x < \frac{1}{2}\right)$.

- (c) Determine the binomial distribution for which the mean is 4 and variance is 3.
- (d) A random variable X has density function given by

$$f(x) = 2e^{-2x}, x \geq 0$$
$$0, \text{ otherwise}$$

then find the moment generating function.

- (e) If X and Y are independent random variables with characteristic functions $\phi_X(\omega)$ and $\phi_Y(\omega)$ respectively then show that $\phi_{X+Y}(\omega) = \phi_X(\omega)\phi_Y(\omega)$.

3. Answer **any four** parts from the following :

$$5 \times 4 = 20$$

- (a) A bag contains 5 balls. Two balls are drawn and are found to be white. What is the probability that all are white ?

- (b) A random variable X has the function

$$f(x) = \frac{c}{x^2 + 1}, \text{ where } -\infty < x < \infty, \text{ then}$$

- (i) find the value of constant c ;
- (ii) find the probability that X^2 lies between $\frac{1}{3}$ and 1.

- (c) The probability density function of a continuous bivariate distribution is given by the joint density function

$$f(x, y) = x + y, 0 < x < 1, 0 < y < 1$$

$$= 0, \text{ elsewhere}$$

Find $E(X)$, $E(Y)$, $\text{var}(X)$, $\text{var}(Y)$ and $E(XY)$

- (d) A coin is tossed until a head appears. What is the expectation of the number of tossed required ?

- (e) If X is a Poisson distributed random variable with parameter μ , then show that $E(X) = \mu$ and $\text{var}(X) = \mu$.

- (f) If X and Y are two independent random variables then show that

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$$

4. Answer **any four** parts from the following :

$$10 \times 4 = 40$$

- (a) (i) If $A_1, A_2, \dots, A_n$ are n mutually exclusive and exhaustive events, then for any event A , prove that

$$P(A) = \sum_{i=1}^n P(A_i)P(A/A_i) \text{ and}$$

$$P(A_i/A) = \frac{P(A_i)P(A/A_i)}{P(A)}$$

- (ii) A restaurant serves two special dishes, A and B to its customers consisting of 60% men and 40% women, 80% of men order dish A and the rest B. 70% of women order B and the rest A. In what ratio of A to B should the restaurant prepare the two dishes ? 5

- (b) (i) Two random variables X and Y have the following joint probability distribution function : 6

$$f(x, y) = 2 - x - y, 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, \text{ otherwise}$$

Find :

(I) Marginal density function

(II) $E(X)$ and $E(Y)$

(III) Conditional density function

- (ii) Determine the Binomial distribution for which the mean is 4 and variance is 3 and find its mode. 4

(c) (i) Find the median of a normal distribution. 5

(ii) A random variable X has density functions given by

$$f(x) = 2e^{-2x}, x \geq 0$$
$$0, x < 0$$

Find (I) mean with the help of moment generating function

(II) $P[|X - \mu| > 1]$. 5

(d) (i) A function $f(x)$ of x is defined as follows :

$$f(x) = 0 \quad \text{for } x < 2$$
$$= \frac{1}{18}(3 + 2x) \text{ for } 2 \leq x \leq 4$$
$$= 0, \quad \text{for } x > 4$$

Show that it is a density function. Also find the probability that a variate with this density will lie in the interval $2 \leq x \leq 3$. 5

(ii) A random variable X can assume values 1 and -1 with probability $\frac{1}{2}$ each. Find -

(I) moment generating function

(II) characteristic function. 5

(e) (i) Derive Poisson distribution as a limiting case of binomial distribution. 5

(ii) If 3% of electric bulbs manufactured by a company are defective, using Poisson's distribution, find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective. [Given $e^{-3} = 0.04979$]. 5

(f) (i) Prove that variance of a random variable X can be expressed as the sum of the expectation of the conditional variance and the variance of the conditional expectation is

$$\text{var}(X) = E[\text{var}(X/Y)] + \text{var}[E(X/Y)].$$

5

(ii) If X is a discrete random variable having probability mass function

Mass Point	0	1	2	3	4	5	6	7
$P(X = x)$	0	k	$2k$	$3k$	$4k$	k^2	$2k^2$	$7k^2 + k$

determine :

(a) k

(b) $P(X < 6)$ and

(c) $P(X \geq 6)$

5

- (g) (i) Prove that independent variables are uncorrelated. With the help of an example show that converse is not true. 5

- (ii) The coefficient of regression of Y on X is $b_{xy} = 1.2$.

If $U = \frac{X - 100}{2}$ and $V = \frac{Y - 200}{3}$

find b_{VU} . 5

- (h) (i) What are the chief characteristics of the normal distribution and normal curve? 4

- (ii) Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1 - x^2) & , -1 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

- (a) What is the value of c ? 3

- (b) Find the cumulative distribution function of X . 3