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3 (Sem–4/CBCS) MAT HC 3

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1 \times 10 = 10$

   (a) Give an example to show that for two non-zero elements a and b of a ring R, the equation $ax = b$ can have more than one solution.

   (b) How many nilpotent elements have in an integral domain?

(c) Which of the following statements is not true?

(i) $\langle 5 \rangle$ is a prime ideal of Z.

(ii) $\langle 5 \rangle$ is a maximal ideal of Z.

(iii) $\langle 5 \rangle$ is a maximal ideal of Z_{20}.

(iv) $\dfrac{Z}{5Z}$ is an integral domain.

(d) Define prime ideal of a ring.

(e) Give example of a commutative ring without zero divisor that is not an integral domain.

(f) Consider the polynomial

$f(x) = 4x^3 + 2x^2 + x + 4$ and

$g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4$ in Z_5.

Compute $f(x) + g(x)$.

(g) Write $f(x) = x^3 + x^2 + x + 1 \in Z_2[x]$ as a product of irreducible polynomial over Z_2.

(h) Which of the following is a primitive polynomial?

(i) $2x^3 + 4x^2 + 6x + 10$

(ii) $5x^2 - 30x - 20$

(iii) $2x^4 + 3x^3 + 5x^2 - 7x + 11$

(iv) $3x^2 - 3x + 3$

(i) State whether the following statement is true or false :

"A polynomial $f(x)$ in $Z[x]$ which is reducible over Z is also reducible over Q."

(j) Choose the correct statement :

(i) Every Euclidean domain is a unique factorization domain.

(ii) Every principal ideal domain is a Euclidean domain.

(iii) Every unique factorization domain is a Euclidean domain.

(iv) Every unique factorization domain is a principal ideal domain.

2. Answer the following questions : 2×5=10

(a) If a and b are *two* idempotents in a commutative ring, then show that $a+b-ab$ is also an idempotent element.

(b) Show that every non-zero element of Z_n is a unit or a zero divisor.

(c) Show that every ring homomorphism $f: Z_n \to Z_n$ is of the form $f(x)=ax$ where $a=a^2$.

(d) Find the zeros of $f(x)=x^2+3x+2$ in Z_6.

(e) Let D be an integral domain and $a,b \in D$. If $\langle a\rangle = \langle b\rangle$, then show that a and b are associates.

3. Answer ***any four*** questions : 5×4=20

(a) The operations $\oplus$ and $\otimes$ defined on the set Z of integers by $a \oplus b = a+b-1$ and $a \otimes b = a+b-ab$. Show that $(Z, \oplus, \otimes,)$ is a ring with unity.

(b) Find all ring homomorphism from $Z \oplus Z$ to Z.

(c) Let R be a commutative ring with unity. Show that an ideal A of R is prime if and only if the quotient ring $\frac{R}{A}$ is an integral domain.

(d) Define principal ideal domain. Show that if F is a field, then $F[x]$ is a principal ideal domain. 1+4=5

(e) Show that every Euclidean domain is a principal ideal domain.

(f) Show that the number of reducible polynomials over Z_p of the form x^2+ax+b is $\frac{p(p+1)}{2}$.

4. Answer the following questions : 10×4=40

(a) (i) Let R be a commutative ring with unity. Show that the set

$$R[x] = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \,/\, a_i \in R,$$

n is a non-negative integer$\}$ is a ring. Also show that if R is an integral domain, then $R[x]$ is also an integral domain. 5+2=7

(ii) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $Z_7[x]$.

Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$. 3

Or

(i) Show that

$Z\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} : a, b \in Z\right\}$ and

$H = \left\{\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} / a, b \in Z\right\}$

are isomorphic as ring. 4

(ii) If *a*, *b* be *any two* ring elements and *m* and *n* be *any two* integers, then show that $(m.a)(n.b) = (mn).(ab)$ 6

(b) (i) Define maximal ideal of a ring. Let *A* be an ideal of a commutative ring with unity *R*. Prove that $\frac{R}{A}$ is a field if and only if *A* is maximal. 1+6=7

(ii) Let *R* be a commutative ring and *A* be any subset of *R*. Show that the nil-radical of *A*,

$N(A) = \left\{r \in R / r^n \in A \text{ for some } n \in N\right\}$

is an ideal of *R*. 3

Or

(i) Let φ be a ring homomorphism from *R* to *S*. Then the mapping from $\frac{R}{\ker(\varphi)}$ to $\varphi(R)$, given by $r + \ker(\varphi) \to \varphi(r)$ is an isomorphism, i.e., $\frac{R}{\ker(\varphi)} \approx \varphi(R)$. 6

(ii) Let ϕ be a ring homomorphism from a ring *R* to a ring *S*. Let *B* be an ideal of *S*. Then $\phi^{-1}[B] = \{r \in R : \phi(r) \in B\}$ is an ideal of *R*. 4

(c) If *F* is a field and $p(x) \in F[x]$, then prove that $\frac{F[x]}{\langle p(x) \rangle}$ is a field if and only if $p(x)$ is irreducible over *F*.

Or

Let F be a field. If $f(x) \in F[x]$ and degree $f(x)$ is 2 or 3, then $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. Is the result true when degree $f(x)$ is greater then 3 ? Justify. 7+3=10

(d) In a principal ideal domain, show that an element is irreducible if and only if it is prime. Use this result to show that $Z[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in Z\}$ is not a principal ideal domain. 7+3=10

Or

(i) In a principal ideal domain, show that any strictly increasing chain of ideals $I_1 \subset I_2 \subset \ldots$ must be finite in length. 5

(ii) Let ϕ be a onto ring homomorphism from a ring R to a ring S. Then prove that ϕ is an isomorphism if and only if $\ker(\phi) = \{0\}$. 3

(iii) Determine all ring homomorphism from the ring of integers Z to itself. 2
