

Total number of printed pages-8

**3 (Sem-3/CBCS) MAT HC 2**

**2023**

**MATHEMATICS**

(Honours Core)

Paper : MAT-HC-3026

**(Group Theory-1)**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer the following questions as directed :

1×10=10

(a) Define order of an element of a group.

(b) In the group  $Q^*$  of all non-zero rational numbers under multiplication, list the

elements of  $\left\langle \frac{1}{2} \right\rangle$ .

(c) Find elements  $A, B, C$  in  $D_4$  such that  $AB = BC$  but  $A \neq C$ .

**Contd.**

- (d) Define simple group.
- (e) State Cauchy's theorem on finite Abelian group.
- (f) State whether the following statement is true **or** false:  
"If  $H$  is a subgroup of the group  $G$  and  $a \in G$ , then  $Ha = \{ha : a \in G\}$  is also a subgroup of  $G$ ."
- (g) Write the order of the alternating group  $A_n$  of degree  $n$ .
- (h) Give an example of an onto group homomorphism which is not an isomorphism.
- (i) State whether the following statement is true **or** false :  
"If the homomorphic image of a group is Abelian then the group itself is Abelian."
- (j) Which of the following statement is true ?
- (a) A homomorphism from a group to itself is called monomorphism.
- (b) A one-to-one homomorphism is called epimorphism.

(c) An onto homomorphism is called endomorphism.

(d) None of the above

2. Answer the following questions :  $2 \times 5 = 10$

(a) In  $D_3$ , find all elements  $X$  such that  $X^3 = X$ .

(b) Consider the group  $Z_2$  under  $+_2$  and  $Z_3$  under  $+_3$ . List the elements of  $Z_2 \oplus Z_3$  and find  $|Z_2 \oplus Z_3|$ .

(c) Express  $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 1 & 4 & 3 & 2 \end{pmatrix}$  as product of transposition and find its order.

(d) If  $\psi : G \rightarrow G'$  is a group homomorphism and  $e$  and  $e'$  be the identity elements of the group  $G$  and  $G'$  respectively then show that  $\psi(e) = e'$ .

(e) Show that in a group  $G$ , if the map  $f : G \rightarrow G'$  defined by  $f(x) = x^{-1}$ ,

$\forall x \in G$  is a homomorphism then  $G$  is Abelian.

3. Answer **any four** questions : (c)  $5 \times 4 = 20$

✓(a) Let  $G$  be a group and  $H$  be a non-empty finite subset of  $G$ . Prove that  $H$  is a subgroup of  $G$  if and only if  $H$  is closed under the operation in  $G$ .

(b) If  $a$  is an element of order  $n$  in a group and  $k$  is a positive integer then prove that

$$\langle a^k \rangle = \langle a^{\gcd(n, k)} \rangle \text{ and}$$

$$|a^k| = \frac{n}{\gcd(n, k)}.$$

✓(c) Show that a subgroup  $H$  of a group  $G$  is a normal subgroup of  $G$  if and only if product of two right cosets of  $H$  in  $G$  is again a right coset of  $H$  in  $G$ .

(d) If  $a, n$  are two integers such that  $n \geq 1$  and  $\gcd(a, n) = 1$ , then prove that  $a^{\phi(n)} \equiv 1 \pmod{n}$ , where  $\phi(n)$  is the Euler's phi function.

(e) Show that any finite cyclic group of order  $n$  is isomorphic to  $\frac{\mathbb{Z}}{\langle n \rangle}$ , where  $\mathbb{Z}$  is the additive group of integers and  $\langle n \rangle = \{0, n, 2n, \dots\}$ .

✓ (f) Let  $\sigma : G \rightarrow \bar{G}$  be a group homomorphism and  $a, b \in G$ .

(i) Show that

$$\sigma(a) = \sigma(b) \Leftrightarrow a \ker \sigma = b \ker \sigma.$$

(ii) If  $\sigma(g) = g'$  then show that

$$\sigma(g') = \{x \in G : \sigma(x) = g'\} = g \ker \sigma.$$

answer **either (a) or (b)** from the following questions :  $10 \times 4 = 40$

✓ (a) Describe the elements of  $D_4$ , the symmetries of a square. Write down a complete Cayley's table for  $D_4$ . Show that  $D_4$  forms a group under composition of functions. Is  $D_4$  an Abelian group?  $2+3+4+1=10$

✓ (b) Prove that every subgroup of a cyclic group is cyclic. Also show that if  $|\langle a \rangle| = n$ , then the order of any subgroup of  $\langle a \rangle$  is a divisor of  $n$ .

Moreover, show that the group  $\langle a \rangle$  has

exactly one subgroup  $\langle a^{\frac{n}{k}} \rangle$  of order  $k$ .

Find the subgroup of  $Z_{30}$  which is of order 3.  $4+2+3+1=10$

5. (a) Show that every quotient group of a cyclic group is cyclic. Give example to show that converse of this statement is

not true in general. Find  $\frac{\mathbb{Z}}{N}$  where  $\mathbb{Z}$  is

the additive group of integers and

$$N = \{5n : n \in \mathbb{Z}\}. \quad 4+3+3=10.$$

(b) (i) Show that every finite group can be represented as a permutation group. 7

(ii) Let  $\phi : G \rightarrow \bar{G}$  be a group homomorphism and  $H$  be a subgroup of  $G$ . If  $\bar{K}$  is a normal subgroup of  $\bar{G}$  then show that  $\phi^{-1}[\bar{K}] = \{k \in G : \phi(k) \in \bar{K}\}$  is a normal subgroup of  $G$ . 3

6. (a) (i) State and prove Lagrange's theorem for the order of subgroup of a finite group. Is the converse true? Justify your answer.

$$1+5+1=7$$

(ii) List the elements of  $\frac{\mathbb{Z}}{4\mathbb{Z}}$  and construct a Cayley's table for it.

3

(b) (i) Show that any two disjoint cycles commute. 5

(ii) Let  $G$  be a group and  $Z(G)$  be the center of  $G$ . If  $\frac{G}{Z(G)}$  is cyclic then show that  $G$  is Abelian. 5

7. (a) Let  $G$  be a group and  $H$  be any subgroup of  $G$ . If  $N$  is any normal subgroup of  $G$ , then show that :

(i)  $H \cap N$  is a normal subgroup of  $H$ .

(ii)  $N$  is a normal subgroup of  $HN$ .

(iii)  $\frac{HN}{N} \cong \frac{H}{H \cap N}$ .

2+2+6=10

(b) Let  $f : G \rightarrow G'$  be an onto group homomorphism and  $H$  be a subgroup of  $G$ ,  $H'$  a subgroup of  $G'$ . Prove that:

(i)  $f[H]$  is a subgroup of  $G'$ .

(ii)  $f^{-1}[H']$  is a subgroup of  $G$  containing  $K = \ker f$ , where

$$f^{-1}[H'] = \{x \in G : f(x) \in H'\}.$$

(iii) There exists a one-to-one correspondence between the set of subgroups of  $G$  containing  $K$  and set of subgroups of  $G'$ .

$$2+3+5=10$$