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3 (Sem-5/CBCS) MAT HE 1/2/3

2023

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION-A

Paper : MAT-HE-5016

(Number Theory)

OPTION-B

Paper : MAT-HE-5026

(Mechanics)

OPTION-C

Paper : MAT-HE-5036

(Probability and Statistics)

Full Marks : 80

Time : Three hours

***The figures in the margin indicate
full marks for the questions.***

Contd.

OPTION-A

Paper : MAT-HE-5016

(Number Theory)

1. Answer the following questions as directed:
1×10=10

(a) Which of the following Diophantine equations cannot have integer solutions ?

(i) $33x + 14y = 115$

(ii) $14x + 35y = 93$

(b) State whether the following statement is true **or** false :

“If a and b are relatively prime positive integers, then the arithmetic progression $a, a + b, a + 2b, \dots$ contains infinitely many primes.”

(c) For any $a \in \mathbb{Z}$ prove that $a \equiv a \pmod{m}$, where m is a fixed integer.

(d) Under what condition the k integers a_1, a_2, \dots, a_k form a CRS \pmod{m} ?

(e) Find $\sigma(p)$ where p is a prime number.

(f) Define Euler's phi function.

(g) If $n = 12789$, find $\tau(n)$.

(h) If x is a real number then show that $[x] \leq x < [x] + 1$, where $[]$ represents the greatest integer function.

(i) Calculate the exponent of the highest power of 5 that divides $1000!$

(j) When an arithmetic function f is said to be multiplicative ?

2. Answer the following questions : 2×5=10

(a) Show that there is no arithmetic progression $a, a + b, a + 2b, \dots$ that consists solely of prime numbers.

(b) Use properties of congruence to show that 41 divides $2^{20} - 1$.

(c) Let $p > 1$ be a positive integer having the property that $p \mid ab \Rightarrow p \mid a$ or $p \mid b$, then prove that p is a prime.

(d) If a is a positive integer and q is its least positive divisor then show that $q \leq \sqrt{a}$.

(e) For $n \geq 3$, evaluate $\sum_{k=1}^n \mu(k!)$, here μ is the Mobius function.

3. Answer **any four** questions : $5 \times 4 = 20$

(a) If $(m, n) = 1$ and $S_1 = \{x_0, x_1, x_2, \dots, x_{m-1}\}$ is a CRS (mod m) and

$S_2 = \{y_0, y_1, y_2, \dots, y_{n-1}\}$ is a CRS (mod n) then show that the set

$S = \{nx_i + my_j : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ form a CRS (mod mn).

(b) Find all integers that satisfy simultaneously

$$x \equiv 5 \pmod{18}; x \equiv -1 \pmod{24};$$

$$x \equiv 17 \pmod{33}$$

(c) If $n \geq 1$ is an integer then show that $\sigma(n)$ is odd if and only if n is a perfect square or twice a perfect square.

(d) If a_1, a_2, \dots, a_k form a RRS (mod m) ie. Reduced Residue System modulo m then show that $k = \phi(m)$.

(e) If x and y be real numbers then show that $[x+y] = [x] + [y]$ and $[-x-y] = [-x] + [-y]$ if and only if one of x or y is an integer.

(f) For $n > 2$, show that $\phi(n)$ is an even integer. Here, ϕ is the Euler phi function.

Answer **either (a) or (b)** from each of the following questions : $10 \times 4 = 40$

4. (a) (i) Show that every positive integer can be expressed as a product of primes. Also show that apart from the order in which prime factors occur in the product, they are unique. $3+4=7$

(ii) If k integers a_1, a_2, \dots, a_k form a CRS (mod m), then show that $m = k$. 3

(b) (i) Show that any natural number greater than one must have a prime factor. 5

- (ii) Prove that if all the $n > 2$ terms of the arithmetic progression $p, p+d, p+2d, \dots, p+(n-1)d$ are prime numbers, then the common difference d is divisible by every prime $q < n$. 5

5. (a) State and prove Wilson's theorem. Is the converse also true? Justify your answer.

1+6+3=10

- (b) Let a and $m > 0$ be integers such that $(a, m) = 1$, then show that $a^{\phi(m)} \equiv 1 \pmod{m}$, here ϕ is the Euler's phi function. Deduce from it the Fermat's Little theorem. Also find the last two digits of 3^{1000} .

5+2+3=10

6. (a) For each positive integer $n \geq 1$, show that

$$\phi(n) = \sum_{d/n} \mu(d) \frac{n}{d} = n \prod_{p/n} \left(1 - \frac{1}{p}\right)$$

- (b) (i) If f and g are two arithmetic functions, then show that the following conditions (A) and (B) are equivalent 7

$$(A) f(n) = \sum_{d/n} g(d)$$

$$(B) g(n) = \sum_{d/n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d/n} \mu\left(\frac{n}{d}\right) f(d)$$

- (ii) If f is a multiplicative arithmetic function, then show that

$$g_1(n) = \sum_{d/n} f(d) \text{ and}$$

$$g_2(n) = \sum_{d/n} \mu(d) f(d) \text{ are both}$$

multiplicative arithmetic functions. 3

7. (a) State and prove Chinese Remainder theorem. 2+8=10

- (b) (i) For $n > 1$, show that the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2}n\phi(n)$. 5

(ii) If $n \geq 1$ is an integer then show that

$$\prod_{d/n} d = n^{\frac{\tau(n)}{2}}. \text{ Is } \prod_{d/n} d \text{ an integer}$$

when $\tau(n)$ is odd? Justify. 5.

OPTION-B

Paper : MAT-HE-5026

(Mechanics)

1. Answer the following questions : $1 \times 10 = 10$
- (a) What is the resultant of two equal forces acting at an angle 120° ?
 - (b) State Lami's theorem.
 - (c) State the principle of conservation of linear momentum.
 - (d) When two parallel forces cannot be compounded into a single resultant force?
 - (e) Define impulsive force with an example.
 - (f) State a necessary and sufficient condition for a system of coplanar forces acting on a rigid body to maintain equilibrium.
 - (g) Define amplitude and frequency of a simple harmonic motion (SHM).
 - (h) Write down the relation between the angle of friction and co-efficient of friction.

- (i) State Newton's that law of motion which defines force as the agent of motion change.
- (j) What is the graphical representation of the moment of a force?

2. Answer the following questions : $2 \times 5 = 10$

- (a) Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another.
- (b) Find the position of centre of gravity (C.G) of a uniform semicircular arc of radius a .
- (c) Prove that earth's gravitational field is a conservative force field.
- (d) Two men have to carry a block of stone of weight 70kg on a light plank. How must the block be placed so that one of the men should bear a weight of 10kg more than the other?
- (e) Prove that the change in kinetic energy of a body is equal to the work done by the acting force.

3. Answer the following questions : **(any four)**
 $5 \times 4 = 20$

- (a) Two forces P and Q acting on a particle at an angle α have a resultant $(2k+1)\sqrt{P^2+Q^2}$. When they act at an angle $90^\circ - \alpha$, the resultant becomes $(2k-1)\sqrt{P^2+Q^2}$, prove that

$$\tan \alpha = \frac{k-1}{k+1}$$

- (b) If the two like parallel forces P and Q acting on a rigid body at A and B be interchanged in position, then show that the point of application of the resultant will be displaced along \overline{AB} through a distance d where
- $$d = \frac{P-Q}{P+Q} \cdot AB \quad (P > Q)$$

- (c) Forces of magnitudes $1, 2, 3, 4, 2\sqrt{2}$ act respectively along the sides $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ and the diagonal \overline{AC} of the square $ABCD$. Show that their resultant is a couple, and find its moment.

- (d) A particle moves towards a centre of attraction starting from rest at a distance a from the centre. If its velocity when at any distance x from the centre

vary as $\sqrt{\frac{a^2 - x^2}{x^2}}$, find the law of force.

- (e) An elastic string without weight, of which the unstretched length is l and the modulus of elasticity is the weight of n ozs, is suspended by one end, and a mass of m ozs. is attached to the other; show that the time of a vertical

oscillation is $2\pi\sqrt{\frac{ml}{ng}}$.

- (f) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained

by the particle is $\frac{V^2}{g}[\lambda - \log(1 + \lambda)]$,

where V is the terminal velocity of the particle and λV is the initial vertical velocity.

4. Answer the following questions : (**any four**)
10×4=40

- (a) (i) Forces P, Q, R acting along $\overline{IA}, \overline{IB}, \overline{IC}$, where I is the in-centre of the triangle ABC , are in equilibrium. Show that. 4

$$P : Q : R = \cos \frac{1}{2}A : \cos \frac{1}{2}B : \cos \frac{1}{2}C$$

- (ii) Forces L, M, N act along the sides of the triangle formed by the lines $x + y - 1 = 0, x - y + 1 = 0, y = 2$. Find the magnitude and the line of action of the resultant. 6

- (b) A body is resting on a rough inclined plane of inclination α to the horizon, the angle of friction being λ ($\alpha > \lambda$). If the body is acted on by a force P , then find the magnitude of P when

(i) the body is just on the point of slipping down.

(ii) the body is just on the point of sliding up.

- (c) (i) Find the C.G of the area of the cardioid $r = a(1 + \cos \theta)$ 5

(ii) Find the C.G. of the solid formed by the revolution of the quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis. 5

(d) (i) Three forces P, Q, R act in the same sense along the sides $\overline{BC}, \overline{CA}, \overline{AB}$ of a triangle ABC . Show that, if their resultant passes through the centroid, then $P \operatorname{Cosec} A + Q \operatorname{Cosec} B + R \operatorname{Cosec} C = 0$ 5

(ii) Forces P, Q, R, S act along the sides $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ of the cyclic quadrilateral $ABCD$, taken in order, where A and B are the extremities of a diameter. If they are in equilibrium, then prove that $R^2 = P^2 + Q^2 + S^2 + \frac{2PQS}{R}$ 5

(e) The velocities of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu \theta$. Find the path. Also show that the accelerations along and perpendicular to the radius vector

$$\text{are } \lambda^2 r - \frac{\mu^2 \theta^2}{r} \text{ and } \mu \theta \left(\lambda + \frac{\mu}{r} \right).$$

(f) A particle moves in a straight line OA with an acceleration which is always directed towards O and varies inversely as the square of its distance from O . If initially the particle were at rest at A , show that the time taken by it to arrive

$$\text{at the origin is } \frac{\pi a^{3/2}}{2\sqrt{2\mu}}.$$

(g) Show that the accelerations along the tangent and the normal to the path of a particle are $\frac{d^2 s}{dt^2} (= v \frac{dv}{ds})$ and $\frac{v^2}{\rho}$, where ρ is the radius of curvature of the curve at the point considered.

(h) A particle falls under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity. Discuss the motion, if the particle starts from rest.

OPTION-C

Paper : MAT-HE-5036

(Probability and Statistics)

1. Answer the following questions : $1 \times 10 = 10$

(a) If A and B are mutually exclusive then find $P(A \cap B)$ and $P(A \cup B)$.

(b) Define probability mass function for discrete random variable.

(c) If $P(x) = \frac{x}{15}, x = 1$
 0 , elsewhere

Find $P\{x = 1 \text{ or } x = 2\}$

(d) If X_1 and X_2 are independent random variables then what will be the modified statement of

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2\text{cov}(X_1, X_2)$$

(e) If a non-negative real valued function f is the probability density function of some continuous random variable, then

what is the value of $\int_{-a}^a f(x) dx$?

(f) Name the discrete distribution for which mean and variance have the same value. What is the value ?

(g) What is meant by mathematical expectation of a random variable ?

(h) Under what condition the binomial distribution becomes the normal distribution.

(i) Write the equation of line of regression of y on x .

(j) State weak law of large number.

2. Answer the following questions : $2 \times 5 = 10$

(a) If the events A and B are independent of A and B separately, is it necessary that they are independent of $A \cap B$? Justify.

(b) Let X be a random variable with the following probability distribution :

$x:$	-3	6	3
$P(X = x):$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find $E(X^2)$

(c) State two properties of Poisson distribution.

- (d) If X is a continuous random variable whose probability density function is given by

$$f(x) = c(4x - 2x^2), 0 < x < 2$$

$$= 0, \text{ otherwise}$$

then find the value of c .

- (e) If X is a random variable, then prove that $\text{Var}(ax + b) = a^2 \text{Var}(X)$ where a and b are constants.

3. Answer **any four** parts from the following :
5×4=20

- (a) A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give 4 white and the second to give 4 black balls if the balls are not replaced before the second draw.

- (b) The probability density function of a two dimensional random variable (X, Y) is given by

$$f(x, y) = x + y, 0 < x + y < 1$$

$$= 0, \text{ elsewhere}$$

Evaluate $P\left(X < \frac{1}{2}, Y > \frac{1}{4}\right)$

- (c) A die is tossed twice. Getting 'a number greater than 4' is considered a success. Find the mean and variance of the probability distribution of the number of success.

- (d) The joint density function of two random variables X and Y is given by

$$f(x, y) = \frac{xy}{96}, 0 < x < 4, 1 < y < 5$$

$$= 0, \text{ otherwise}$$

Find

(i) $E(X)$

(ii) $E(Y)$

(iii) $E(2X + 3Y)$

- (e) For any two independent random variable X and Y , for which $E(X)$ and $E(Y)$ exists, show that

$$E(XY) = E(X)E(Y)$$

- (f) With usual notation for a binomial variate X , given that $9p(X=4) = p(X=2)$ when $n = 6$

Find the value of p and q .

4. Answer **any four** parts from the following:
10×4=40

(a) (i) If A and B are any two events and are not disjoint then show that
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
Hence find $P(A \cup B \cup C)$.

(ii) From a bag containing 4 white and 6 red balls, three balls are drawn at random. Find the expected number of white balls drawn.

(b) (i) The joint density function of x and y is given by

$$f(x, y) = 2e^{-x}e^{-2y}, 0 < x < \alpha, 0 < y < \alpha$$

$$0, \text{ otherwise}$$

compute $P(X > 1, Y < 1)$, $P(X < Y)$ and $P(X < \alpha)$.

(ii) If X is a random Poisson variate with parameter m , then show that

$$p(X \geq n) - p(X \geq n+1) = \frac{e^{-m} m^n}{L^n}$$

(c) (i) If X is a binomial variate then prove that

$$\text{Cov}\left(\frac{X}{n}, \frac{n-X}{n}\right) = -\frac{pq}{n}$$

(ii) Show that normal distribution may be regarded as a limiting case of Poisson's distribution on the parameter $m \rightarrow \infty$.

(d) (i) Prove that the moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating function.

(ii) Define moments and moment generating function of a random variable X . If $M(t)$ is the moment generating function of a random variable X about the origin, show that the moment μ'_r is given by

$$\mu'_r = \left[\frac{d^r M(t)}{dt^r} \right]_{t=0}$$

(e) (i) If $U = \frac{X-a}{h}$, $V = \frac{Y-b}{k}$ where a, b, h, k are constants, $h > 0, k > 0$ then show that $r(X, Y) = r(U, V)$.

(r represents the correlation co-efficient)

- (ii) The two regression equations of the variables x and y are

$$x = 19.13 - 0.87y$$

$$y = 11.64 - 0.50x$$

Find (l) mean of x 's

(m) mean of y 's

(n) correlation co-efficients between x and y .

- (f) (i) Find the mean and variance of a Binomial distribution.

- (ii) If X is a random variable with mean μ and variance σ^2 then for any positive number k , prove that

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

- (g) (i) A function $f(x)$ of x is defined as follows :

$$\begin{aligned} f(x) &= 0 && \text{for } x < 2 \\ &= \frac{1}{18}(3 + 2x) && \text{for } 2 \leq x \leq 4 \\ &= 0 && \text{for } x > 4 \end{aligned}$$

Show that it is a density function. Also find the probability that a variate with this density will lie in the interval $2 \leq x \leq 3$.

- (ii) Two random variables X and Y have the following joint probability distribution function.

$$f(x, y) = \begin{cases} 2 - x - y, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (l) marginal density function

(m) $E(X)$ and $E(Y)$

(n) conditional density function

- (h) (i) Show that Poisson distribution is a limiting case of the Negative Binomial Distribution.

- (ii) Let the random variable X_i assume values i and $-i$ with equal probabilities. Show that the law of large number cannot be applied to the independent variables $X_1, X_2, X_3, \dots, X_n$.