

6. Answer **either** (a) and (b) **or** (c) and (d) of the following questions : 10

(a) Suppose that

$$z_n = x_n + iy_n \quad (n = 1, 2, 3 \dots) \text{ and}$$

$$S = X + iY. \text{ Prove that}$$

$$\sum_{n=1}^{\infty} z_n = S \text{ if and only if}$$

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y. \quad 5$$

(b) Find the Maclaurin series for the entire function  $f(z) = \sin z$ . 5

**Or**

(c) Define absolutely convergent series. Prove that the absolute convergence of a series of complex numbers implies the convergence of the series. 1+3=4

(d) Find the Maclaurin series for the entire function  $f(z) = \cos z$ . 6

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**3 (Sem-6/CBCS) MAT HC 1 (N/O)**

**2023**

**MATHEMATICS**

(Honours Core)

Paper : MAT-HC-6016

**(New Syllabus/Old Syllabus)**

Full Marks : 80/60

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

**New Syllabus**

Full Marks : 80

**(Riemann Integration and Metric Spaces)**

Answer the following as directed :

1×10=10

(a) Define the discrete metric  $d$  on a non-empty set  $X$ .

(b) Let  $F_1$  and  $F_2$  be two subsets of a metric space  $(X, d)$ . Then

(i)  $\overline{F_1 \cup F_2} = \overline{F_1} \cap \overline{F_2}$

(ii)  $\overline{F_1 \cap F_2} = \overline{F_1} \cup \overline{F_2}$

(iii)  $\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$

(iv)  $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$

(Choose the correct option)

(c) Let  $(X, d)$  be a metric space and  $A \subset X$ . Then

(i)  $\text{Int} A$  is the largest open set contained in  $A$ .

(ii)  $\text{Int} A$  is the largest open set containing  $A$ .

(iii)  $\text{Int} A$  is the intersection of all open sets contained in  $A$ .

(iv)  $\text{Int} A = A$

(Choose the correct option)

(d) Let  $(X, d)$  be a disconnected metric space.

We have the statements :

I. There exists two non-empty disjoint subsets  $A$  and  $B$ , both open in  $X$ , such that  $X = A \cup B$ .

II. There exists two non-empty disjoint subsets  $A$  and  $B$ , both closed in  $X$ , such that  $X = A \cup B$ .

(i) Only I is true

(ii) Only II is true

(iii) Both I and II are true

(iv) None of I and II is true

(Choose the correct option)

(e) Find the limit points of the set of rational numbers  $Q$  in the usual metric  $R_u$ .

(f) In a metric space, the intersection of infinite number of open sets need not be open. Justify it with an example.

(g) Define a mapping  $f : X \rightarrow Y$ , so that the metric spaces  $X = [0, 1]$  and  $Y = [0, 2]$  with usual absolute value metric are homeomorphic.

(h) Define Riemann sum of  $f$  for the tagged partition  $(P, t)$ .

(i) State the first fundamental theorem of calculus.

(j) Examine the existence of improper Riemann integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

2. Answer the following questions :  $2 \times 5 = 10$

(a) Prove that in a metric space  $(X, d)$  every open ball is an open set.

(b) Prove that the function  $f : [0, 1] \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$  is an uniformly continuous mapping.

(c) Let  $d_1$  and  $d_2$  be two metrics on a non-empty set  $X$ . Prove that they are equivalent if there exists a constant  $K$  such that

$$\frac{1}{K} d_2(x, y) \leq d_1(x, y) \leq K d_2(x, y)$$

(d) If  $m$  is a positive integer, prove that  $\overline{m+1} = m!$

(e) Let  $f(x) = x$  on  $[0, 1]$ .

$$\text{Let } P = \left\{ x_i = \frac{i}{4}, i = 0, \dots, 4 \right\}$$

Find  $L(f, P)$  and  $U(f, P)$ .

3. Answer the following questions (**any four**):  
 $5 \times 4 = 20$

(a) Let  $(X, d)$  be metric space and  $F$  be a subset of  $X$ . Prove the  $F$  is closed in  $X$  if and only if  $F^c$  is open.

(b) Define diameter of a non-empty bounded subset of a metric space  $(X, d)$ . If  $A$  is a subset of a metric space  $(X, d)$ , then prove that  $d(A) = d(\overline{A})$ .

$1 + 4 = 5$

(c) Let  $(X, d)$  be a metric space. Then prove that the following statements are equivalent :

(i)  $(X, d)$  is disconnected.

(ii) There exists two non-empty disjoint subsets  $A$  and  $B$ , both open in  $X$ , such that  $X = A \cup B$ .

(d) Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be integrable functions. Then prove that  $f + g$  is integrable and

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

(e) Discuss the convergence of the integral

$$\int_1^\infty \frac{1}{x^p} dx \text{ for various values of } p.$$

(f) Consider  $f : [0, 1] \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$ . Prove that  $f$  is integrable.

4. Answer the following questions :  $10 \times 4 = 40$

(a) (i) Let  $X$  be the set of all bounded sequences of numbers  $\{x_i\}_{i \geq 1}$  such that  $\sup_i |x_i| < \infty$ .

For  $x = \{x_i\}_{i \geq 1}$  and  $y = \{y_i\}_{i \geq 1}$  in  $X$  define  $d(x, y) = \sup_i |x_i - y_i|$ .

Prove that  $d$  is a metric on  $X$ .

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(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. Is the converse true? Justify with an example.  $4 + 1 = 5$

Or

(a) (i) Show that  $d(x, y) = \sqrt{|x - y|}$  defines a metric on the set of reals. 4

(ii) Show that the metric space  $X = \mathbb{R}^n$  with the metric given by  $d_p(x, y) = (\sum |x_i - y_i|^p)^{1/p}$ ,  $p \geq 1$  where  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  are in  $\mathbb{R}^n$  is a complete metric space. 6

(b) (i) Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces and  $f : X \rightarrow Y$ . If  $f$  is continuous on  $X$ , prove the following :  $3 + 3 = 6$

(i)  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for all subsets of  $B$  of  $Y$

(ii)  $f(\overline{A}) \subseteq \overline{f(A)}$  for all subsets  $A$  of  $X$

(ii) Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces and  $f : X \rightarrow Y$  be uniformly continuous. Prove that if  $\{x_n\}_{n \geq 1}$  is a Cauchy sequence in  $X$ , then  $\{f(x_n)\}_{n \geq 1}$  is a Cauchy sequence in  $Y$ . 4

Or

(b) Define fixed point of a mapping  $T: X \rightarrow X$ . Let  $T: X \rightarrow X$  be a contraction of the complete metric space  $(X, d)$ . Prove that  $T$  has a unique fixed point. 2+8=10

(c) (i) Prove that if the metric space  $(X, d)$  is disconnected, then there exists a continuous mapping of  $(X, d)$  onto the discrete two element space  $(X_0, d_0)$ . 5

(ii) Let  $(X, d)$  be a metric space and  $A^0, B^0$  are interiors of the subsets  $A$  and  $B$  respectively. Prove that

$$(A \cap B)^0 = A^0 \cap B^0;$$
$$(A \cup B)^0 \supseteq A^0 \cup B^0. \quad 5$$

Or

(c) (i) When is a non-empty subset  $Y$  of a metric space  $(X, d)$  said to be connected? Let  $(X, d_X)$  be a connected metric space and  $f: (X, d_X) \rightarrow (Y, d_Y)$  be a continuous mapping. Prove that the space  $f(X)$  with the metric induced from  $Y$  is connected. 5

(ii) Let  $(X, d)$  be a metric space and  $Y \subseteq X$ . If  $X$  is separable then prove that  $Y$  with the induced metric is also separable. 5

(d) (i) If  $f$  is Riemann integrable on  $[a, b]$  then prove that it is bounded on  $[a, b]$ . 5

(ii) When is an improper Riemann integral said to exist? Show that the improper integral of  $f(x) = |x|^{-1/2}$  exists on  $[-1, 1]$  and its value is 4. 1+4=5

Or

(d) (i) Let  $f: [a, b] \rightarrow \mathbb{R}$  be integrable. Then prove that the indefinite integral  $F(x) = \int_a^x f(t) dt$  is continuous on  $[a, b]$ .

Further prove that if  $f$  is continuous at  $x \in [a, b]$ , then  $F$  is differentiable at  $x$  and  $F'(x) = f(x)$ . 3+3=6

(ii) Evaluate

$$\lim_{x \rightarrow \infty} \frac{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}{\sqrt{n^3}} = \frac{2}{3} \quad 4$$

## Old Syllabus

Full Marks : 60

### (Complex Analysis)

1. Answer the following as directed :  $1 \times 7 = 7$

(a) Any complex number  $z = (x, y)$  can be written as

(i)  $z = (0, x) + (1, 0)(0, y)$

(ii)  $z = (x, 0) + (0, 1)(y, 0)$

(iii)  $z = (x, 0) + (0, 1)(0, y)$

(iv)  $z = (0, x) + (1, 0)(y, 0)$

(Choose the correct option)

(b) Write the function  $f(z) = z^2 + z + 1$  in the form  $f(z) = u(x, y) + iv(x, y)$ .

(c) The value of  $\lim_{z \rightarrow \infty} \frac{2z+i}{z+1}$  is

(i)  $\infty$

(ii) 0

(iii) 2

(iv)  $i$

(Choose the correct option)

(d) Determine the singular points of the function

$$f(z) = \frac{z^2 + 1}{(z+2)(z^2 + 2z + 2)}$$

(e) Define an analytic function of the complex variable  $z$ .

(f)  $e^{i(2n+1)\pi}$  is equal to

(i) 1

(ii) -1

(iii) 0

(iv) 2

(Choose the correct option)

(g)  $\text{Log}(-1)$  is equal to

(i)  $\frac{\pi}{2}i$

(ii)  $\pi i$

(iii)  $-\frac{\pi}{2}i$

(iv)  $-\pi i$

(Choose the correct option)

2. Answer the following questions :  $2 \times 4 = 8$

(a) Show that  $\lim_{z \rightarrow \infty} \frac{1+z^2}{z-1} = \infty$

(b) If  $f(z) = e^x \cdot e^{iy} = e^z$  where  $z = x + iy$ , show that  $f'(z) = e^x \cos y + ie^x \sin y$ .

(c) Show that  $\int_C f(z) dz = 0$  when the

contour  $C$  is the unit circle  $|z|=1$  in

either direction and  $f(z) = \frac{z^2}{z-3}$ .

(d) Show that the sequence  $z_n = \frac{1}{n^3} + i$

$(n = 1, 2, 3, \dots)$  converges to  $i$ .

3. Answer **any three** questions from the following :  $5 \times 3 = 15$

(a) If  $z_1$  and  $z_2$  are complex numbers then show that

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

(b) Suppose a function  $f(z)$  be analytic throughout a given domain  $D$ . If  $|f(z)|$  is constant throughout  $D$ , then prove that  $f(z)$  is constant in  $D$ .

(c) Show that the derivative of the real valued function  $f(z) = |z|^2$  exists only at  $z = 0$ .

(d) If a function  $f$  is analytic at a given point, then prove that its derivatives of all orders are analytic there too.

(e) State Cauchy integral formula. Apply it to find  $\int_C \frac{f(z)}{z+i} dz$  where  $f(z) = \frac{z}{9-z^2}$  and  $C$  is the positively oriented circle  $|z|=2$ .

4. Answer **either** (a) and (b) **or** (c) of the following questions : 10

(a) (i) Show that if  $f(z) = \frac{i\bar{z}}{2}$  in the open

disk  $|z| < 1$ , then

$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$$

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(ii) Show that the function

$f(z) = e^{-y} \sin x - i e^{-y} \cos x$  is entire. 3

(b) If a function  $f(z)$  is continuous and nonzero at a point  $z_0$ , then prove that  $f(z) \neq 0$  throughout some neighbourhood of that point. 4

Or

(c) Let the function

$f(z) = u(x, y) + i v(x, y)$  be defined throughout some  $\varepsilon$  neighbourhood of a point  $z_0 = x_0 + i y_0$ , and suppose that

(i) the first order partial derivatives of the functions  $u$  and  $v$  with respect to  $x$  and  $y$  exist everywhere in the neighbourhood;

(ii) those partial derivatives are continuous at  $(x_0, y_0)$  and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x \text{ at } (x_0, y_0).$$

Prove that  $f'(z)$  exists and

$f'(z_0) = u_x + i v_x$  where the right hand side is to be evaluated at  $(x_0, y_0)$ . 10

5. Answer **either** (a) and (b) **or** (c) and (d) of the following questions : 10

(a) Find the value of  $\int_C \bar{z} dz$  where  $C$  is the

right-hand half  $z = 2e^{i\theta}$   $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$  of the circle  $|z| = 2$  from  $z = -2i$  to  $z = 2i$ . 5

(b) Let  $C$  be the arc of the circle  $|z| = 2$  from  $z = 2$  to  $z = 2i$  that lies in the 1st quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7} \quad 5$$

Or

(c) State Liouville's theorem. 1

(d) Prove that any polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n \quad (a_n \neq 0)$$

of degree  $n$  ( $n \geq 1$ ) has at least one zero. 9