3 (Sem–3/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

*The figures in the margin indicate
full marks for the questions.*

1. Answer the following questions :   1×10=10

    (a) Is 0 a cluster point of $(0,1)$?

    (b) " If the limit of a function f at a point
    C of its domain does not exist, then f
    diverges at C."   *(Write True or False)*

    (c) Define $\underset{x \to c}{Lim}\, f(x) = \infty$, where $A \subseteq \mathbb{R}$ and
    $f : A \to \mathbb{R}$ and $C \in \mathbb{R}$ is a cluster point
    of A.

(d) Write sequential criterion for continuity.

(e) What do you mean by an unbounded function on a set ?

(f) Let $A, B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous on A and let $g : B \to \mathbb{R}$ be continuous on B. Under what condition $g \circ f : A \to \mathbb{R}$ is continuous on A ?

(g) "If a function is continuous then it is uniformly continuous."

(Write True **or** False)

(h) If functions $f_1, f_2, \ldots, f_n$ are differentiable at c, write the expression for $(f_1 . f_2 \ldots . f_n)'(c)$.

(i) The function $f(x) = x$ is defined on the interval $I = [0,1]$. Is 0 a relative maximum of f ?

(j) Define Taylor's polynomial for a function f at a point x_0, supposing f has an nth derivative at x_0.

2. Answer the following questions : $2 \times 5 = 10$

(a) Use $\varepsilon - \delta$ definition of limit to show that $\lim\limits_{x \to 0} \dfrac{x^2}{|x|} = 0$.

(b) Show that the absolute value function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(c) Give an example of a function $f : [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$, but $|f|$ is continuous on $[0,1]$.

(d) "Continuity at a point is not a sufficient condition for the derivative to exist at that point." Justify your answer.

(e) Show that $\lim\limits_{x \to 0+} \dfrac{\sin x}{\sqrt{x}} = 0$.

3. Answer **any four** parts : $5 \times 4 = 20$

(a) Prove that a number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence $\{a_n\}$ in A such that $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

(b) Show that (using $\varepsilon - \delta$ definition of limit)

$$\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$$

(c) Prove that if $I = [a, b]$ is a closed bounded interval and if $f : I \to \mathbb{R}$ is continuous on I then f is bounded on I.

(d) Show that if f and g are uniformly continuous on a subset A of $\mathbb{R}$ then $f + g$ is uniformly continuous on A.

(e) Suppose that f is continuous on a closed interval $I = [a, b]$ and that f has a derivative in the open interval (a, b). Then there exists *at least one* point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a).$$

(f) Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then prove that f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

4. Answer *any four* parts :   10×4=40

(a) Prove that a real valued function f is continuous at $c \in \mathbb{R}$ if and only if whenever every sequence $\{c_n\}$, converging to c, then corresponding sequence $\{f(c_n)\}$ converges to $f(c)$.

(b) (i) Show that every infinite bounded subset of $\mathbb{R}$ has *at least one* limit point.   5

(ii) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If $a \leq f(x) \leq b$ for all $x \in A$, $x \neq c$ and if $\lim_{x \to c} f(x)$ exist then prove that

$$a \leq \lim_{x \to c} f \leq b .$$ 5

(c) (i) Let $I = [a, b]$ be a closed bounded interval. Let $f : I \to \mathbb{R}$ be such that the f is continuous. Prove that f is uniformly continuous on $[a, b]$.   5

(ii) Show that the function $f(x) = \dfrac{1}{x^2}$

is uniformly continuous on

$A = [1, \infty)$.   5

(d) Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

(e) (i) Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) = \{ f(x) : x \in I \}$ is a closed bounded interval.   6

(ii) Let $A, B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b = f(c) \in B$, then show that the composition $g \circ f : A \to \mathbb{R}$ is continuous at c.   4

(f) (i) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ or if $f(a) > 0 > f(b)$, then prove that there exists a number $c \in (a, b)$ such that $f(c) = 0$.   6

(ii) Use the definition to find the derivative of the function $f(x) = \dfrac{1}{\sqrt{x}}$

for $x > 0$.   4

(g) (i) State and prove Taylor's theorem.   2+5=7

(ii) Using the Mean Value theorem prove that $\left| \sin x - \sin y \right| \le \left| x - y \right|$ for all x, y in $\mathbb{R}$.   3

(h) (i) Show that
$$1 - \frac{1}{2} x^2 \le \cos x$$
for all $x \in \mathbb{R}$ 5

(ii) Evaluate $\displaystyle \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$ 5

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