OPTION–A

Paper : MAT–HE–5016

(*Number Theory*)

1. Answer the following questions as directed: $1\times10=10$

(a) Which of the following Diophantine equations cannot have integer solutions?

(i) $33x + 14y = 115$

(ii) $14x + 35y = 93$

(b) State whether the following statement is true **or** false :

"If a and b are relatively prime positive integers, then the arithmetic progression a, a + b, a + 2b,... contains infinitely many primes."

(c) For any $a \in Z$ prove that $a \equiv a \pmod{m}$, where m is a fixed integer.

(d) Under what condition the k integers $a_1, a_2, \ldots, a_k$ form a CRS (mod m) ?

(e) Find $\sigma(p)$ where p is a prime number.

(f) Define Euler's phi function.

(g) If $n = 12789$, find $\tau(n)$.

(h) If x is a real number then show that $[x] \leq x < [x]+1$, where [ ] represents the greatest integer function.

(i) Calculate the exponent of the highest power of 5 that divides 1000!

(j) When an arithmetic function f is said to be multiplicative ?

2. Answer the following questions : 2×5=10

(a) Show that there is no arithmetic progression a, $a + b$, $a + 2b$,.... that consists solely of prime numbers.

(b) Use properties of congruence to show that 41 divides $2^{20} - 1$.

(c) Let $p > 1$ be a positive integer having the property that $p/ab \Rightarrow p/a$ or p/b, then prove that p is a prime.

(d) If a is a positive integer and q is its least positive divisor then show that $q \leq \sqrt{a}$.

(e) For $n \geq 3$, evaluate $\sum_{k=1}^{n} \mu(k!)$, here μ is the Mobius function.

3. Answer ***any four*** questions : 5×4=20

(a) If $(m, n) = 1$ and $S_1 = \{x_0, x_1, x_2, ..., x_{m-1}\}$ is a CRS (mod m) and $S_2 = \{y_0, y_1, y_2, ..., y_{n-1}\}$ is a CRS (mod n) then show that the set

$$S = \{nx_i + my_j : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$$

form a CRS (mod mn).

(b) Find all integers that satisfy simultaneously

$$x \equiv 5 (\text{mod } 18);\ x \equiv -1 (\text{mod } 24);$$
$$x \equiv 17 (\text{mod } 33)$$

(c) If $n \geq 1$ is an integer then show that $\sigma(n)$ is odd if and only if n is a perfect square or twice a perfect square.

(d) If $a_1, a_2, ..., a_k$ form a RRS (mod m) ie. Reduced Residue System modulo m then show that $k = \phi(m)$.

(e) If x and y be real numbers then show that $[x+y]=[x]+[y]$ and $[-x-y]=[-x]+[-y]$ if and only if one of x or y is an integer.

(f) For $n>2$, show that $\phi(n)$ is an even integer. Here, ϕ is the Euler phi function.

Answer **either** (a) **or** (b) from each of the following questions : 10×4=40

4. (a) (i) Show that every positive integer can be expressed as a product of primes. Also show that apart from the order in which prime factors occur in the product, they are unique. 3+4=7

(ii) If k integers $a_1, a_2, \ldots, a_k$ form a CRS (mod m), then show that $m=k$. 3

(b) (i) Show that any natural number greater than one must have a prime factor. 5

a, NH, XM (addmore)

(ii) Prove that if all the $n > 2$ terms of the arithmetic progression $p, p+d, p+2d, \ldots, p+(n-1)d$ are prime numbers, then the common difference d is divisible by every prime $q < n$. 5

5. (a) State and prove Wilson's theorem. Is the converse also true ? Justify your answer. 1+6+3=10

(b) Let a and $m > 0$ be integers such that $(a, m) = 1$, then show that $a^{\phi(m)} \equiv 1 \pmod{m}$, here ϕ is the Euler's phi function. Deduce from it the Fermat's Little theorem. Also find the last two digits of 3^{1000}. 5+2+3=10

6. (a) For each positive integer $n \geq 1$, show that

$$\phi(n) = \sum_{d/n} \mu(d)\frac{n}{d} = n\prod_{p/n}\left(1 - \frac{1}{p}\right)$$

(b) (i) If f and g are two arithmetic functions, then show that the following conditions (A) and (B) are equivalent 7

(A) $f(n) = \sum_{d/n} g(d)$

(B) $g(n) = \sum_{d/n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d/n} \mu\left(\frac{n}{d}\right) f(d)$

(ii) If f is a multiplicative arithmetic function, then show that

$g_1(n) = \sum_{d/n} f(d)$ and

$g_2(n) = \sum_{d/n} \mu(d) f(d)$ are both

multiplicative arithmetic functions. 3

7. (a) State and prove Chinese Remainder theorem. 2+8=10

(b) (i) For $n > 1$, show that the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n\phi(n)$. 5

(ii) If $n \geq 1$ is an integer then show that

$$\prod_{d/n} d = n^{\frac{\tau(n)}{2}}$$

. Is $\prod_{d/n} d$ an integer when $\tau(n)$ is odd ? Justify. 5
