2022

MATHEMATICS

(Honours)

Paper : MAT–HC–5026

(Linear Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** questions : 1×10=10

(i) "A plane in $\mathbb{R}^3$ not through the origin is a subspace of $\mathbb{R}^3$."
(State True ***or*** *False)*

(ii) If the equation $AX = 0$ has only the trivial solution then what is the null space of A?

(iii) Suppose two matrices are row equivalent. Are their row spaces the same?

(iv) Let A be matrix of order $m \times n$. When the column space of A and $\mathbb{R}^m$ are equal ?

(v) Is the set $\{sint, cost\}$ linearly independent in $C[0,1]$?

(vi) What is the dimension of zero vector space ?

(vii) If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A ?

(viii) "0 is an eigenvalue of a matrix A if and only if A is invertible."

*(State True **or** False)*

(ix) Let A be an $n \times n$ matrix such that determinant of A is zero. Is A invertible ?

(x) When two matrices A and B are said to be similar ?

(xi) Define complex eigenvalue of a matrix.

(xii) Let an $n \times n$ matrix has n distinct eigenvalues. Is it diagonalizable ?

(xiii) What do you mean by distance between two vectors in $\mathbb{R}^n$?

(xiv) Which vector is orthogonal to every vector in $\mathbb{R}^n$?

(xv) Is inner product of two vector u and v in $\mathbb{R}^n$ commutative ?

(xvi) "An orthogonal matrix is invertible."

*(State True **or** False)*

(xvii) If the number of free variables in the equation $Ax = 0$ is p, then what is the dimension of null space of A ?

(xviii) Let T be a linear operator on a vector space V. Is the subspace of $\{0\}$ of V T-invariant ?

2. Answer ***any five*** questions : 2×5=10

(i) Show that the set H of all points of $\mathbb{R}^2$ of the form $(3r,\ 2+5r)$ is not a vector space.

(ii) Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ and let

NO

$u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$. Is u in null space of A ?

(iii) In $\mathbb{R}^3$, show that the set $W = \{(a, b, c) : a^2 + b^2 + c^2 \leq 1\}$ is not a subset of V.

(iv) Let $P_1(t) = 1, P_2(t) = t, P_3(t) = 4 - t$. Show that $\{P_1, P_2, P_3\}$ is linearly dependent in the vector space of polynomials.

(v) Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$. Examine whether u is a eigenvector of A.

(vi) The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalue of the matrix.

(vii) Show that the eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

(viii) Let $v = (1, -2, 2, 0)$. Find a unit vector u in the same direction as v.

(ix) Let $u = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Compute $\frac{u.v}{u.u}$.

(x) Suppose $S = \{u_1, u_2, \ldots, u_n\}$ contains a dependent subset. Show that S is also dependent.

3. Answer ***any four*** questions : 5×4=20

(i) Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$. Find a non-zero vector in column space of A and a non-zero vector in null space of A.

(ii) If a vector space V has a basis $B = \{b_1, b_2, \ldots, b_n\}$, then prove that any set in V containing more than n vectors must be linearly dependent.

(iii) Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for a vector space V, then prove that the co-ordinate mapping $x \to [x]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

(iv) Prove that similar matrices have the same characteristic polynomial and hence the same eigenvalues.

(v) Is 5 an eigenvalue of $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?

(vi) Let $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$ and $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$. Show that U has orthonormal columns and $\|Ux\| = \|x\|$.

(vii) Find a QR factorization of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(viii) Find the range and kernel of $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ x-y \end{bmatrix}$.

4. Answer ***any four*** questions : 10×4=40

(i) Find the spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -1 \end{bmatrix}.$$

(ii) Let $S = \{v_1, v_2, \ldots, v_r\}$ be a set in a vector space V over $\mathbb{R}$ and let $H = span\,\{v_1, v_2, \ldots, v_r\}$. Prove that—

(a) if one of the vectors in S is a linear combination of the remaining vectors in S, then the set formed from S by removing that vector still spans H;

(b) if $H \neq \{0\}$, some subset of S is a basis for H.

5+5=10

(iii) Let V be the vector space of 2×2 symmetric matrices over $\mathbb{R}$. Show that $dim\ V = 3$. Also find the co-ordinate vector of the matrix

$A = \begin{bmatrix} 4 & -11 \\ -11 & -7 \end{bmatrix}$ relative to the basis

$$\left\{ \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -1 & -5 \end{bmatrix} \right\}.$$

5+5=10

(iv) Define a diagonalizable matrix. Prove that an $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvector. 1+9=10

(v) (a) Show that λ is an eigenvalue of an invertible matrix A if and only if λ^{-1} is an eigenvalue of A^{-1}.

(b) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, then show that $k\lambda_1, k\lambda_2, ..., k\lambda_n$ are the eigenvalues of kA.

(c) Show that the matrices A and A^T (transpose of A) have the same eigenvalues.

5+2½+2½=10

(vi) Compute A^8 where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

(vii) Define orthogonal set and orthogonal basis of $\mathbb{R}^n$. Show that $S = \{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Also express the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vector in S.

(1+1)+5+3=10

(viii) Let V be an inner product space. Show that—

(a) $\langle v, 0 \rangle = \langle 0, v \rangle = 0$;

(b) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$ where $u, v, w \in V$;

(c) Define norm of a vector in V;

(d) For u, v in V, show that $|\langle u, v \rangle| \leq \|u\| \|v\|$.

2+2+1+5=10

(ix) What do you mean by Gram-Schmidt process? Prove that if $\{x_1, x_2, ..., x_p\}$ is a basis for a subspace W or $\mathbb{R}^n$ and define $v_1 = x_1$

$$v_2 = x_2 - \frac{x_2 . v_1}{v_1 . v_2} v_1$$

$$v_3 = x_3 - \frac{x_3 . v_1}{v_1 . v_1} v_1 - \frac{x_3 . v_2}{v_2 . v_2} v_2$$

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$$v_p = x_p - \frac{x_p . v_1}{v_1 . v_1} v_1 - \frac{x_p . v_2}{v_2 . v_2} v_2 - ... \frac{x_p . v_{p-1}}{v_{p-1} . v_{p-1}} v_{p-1}$$

then $\{v_1, v_2, ... v_p\}$ is an orthogonal basis for W. Also if $W = span\{x_1, x_2\}$

where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis $\{v_1, v_2\}$ for W.

1+6+3=10

(x) Define orthogonal complement of a subspace. Let $\{u_1, u_2, ... u_5\}$ be an orthogonal basis for $\mathbb{R}^5$ and $y = c_1u_1 + ... + c_5u_5$. If the subspace $W = span\ \{u_1, u_2\}$ then write y as the sum of vectors Z_1 in W and a vector Z_2 in complement of W. Also find the distance from y to $W = span\ \{u_1, u_2\}$,

where $y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

1+6+3=10