3 (Sem–5/CBCS) STA HC 1

2022

STATISTICS

(Honours)

Paper : STA–HC–5016

(Stochastic Processes and Queueing Theory)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** of the following questions as directed : $1 \times 7 = 7$

   (a) The value of $P(1)$ is

   (i) 0

   (ii) 1

   (iii) ∞

   (iv) None of the above

   (Choose the correct option)

Contd.

(b) The mean of X in terms of the probability generating function (p.g.f.) of X is given by

(i) $P''(1)$

(ii) $P'(s)$

(iii) $P'(1)$

(iv) $P'(0)$

(Choose the correct option)

(c) The p.g.f. of sum of two independent random variables X and Y is the sum of the p.g.f. of X and that of Y.

*(State True **or** False)*

(d) Define state space of a stochastic process.

(e) A process which is not stationary is said to be ________. *(Fill in the blank)*

(f) In an irreducible Markov chain, every state cannot be reached from every other state. *(State True **or** False)*

(g) If $\{N_1(t)\}$ and $\{N_2(t)\}$ are two independent Poisson processes with rates λ_1 and λ_2 respectively then $N_1(t)-N_2(t)$ is a

(i) Poisson process with rate $\lambda_1+\lambda_2$

(ii) Poisson process with rate $\lambda_1-\lambda_2$

(iii) Poisson process with rate λ_1/λ_2

(iv) Not a Poisson process

(Choose the correct option)

(h) A state of a Markov chain is said to be ergodic if it is

(i) persistent non-null and aperiodic state

(ii) transient non-null and aperiodic state

(iii) persistent non-null and periodic state

(iv) transient null and aperiodic state

(Choose the correct option)

(i) Define traffic intensity.

(j) In M/M/1 queueing model, the interarrival time as well as service time follows ________ distribution.

(Fill in the blank)

(k) Define homogeneous Markov chain.

(l) Families of random variables, which are functions of, say, time, are known as ________. *(Fill in the blank)*

2. Answer ***any four*** of the following questions : 2×4=8

(a) Define bivariate probability generating function of a pair of random variables X and Y.

(b) Define transition probability matrix.

(c) State *any two* postulates of Poisson process.

(d) Is Poisson process a stationary process? If not, why?

(e) Differentiate between steady state and transient state of a queueing system.

(f) Distinguish between irreducible and reducible Markov chain.

(g) What are the basic features of a queueing system?

(h) Write *any two* properties of Poisson process.

3. Answer ***any three*** of the following questions: 5×3=15

(a) Let X be a random variable with p.m.f

$$p_k = P_r\{X = k\} = q^k P, \; k = 0, 1, 2, ...$$

$$0 < q = 1 - p < 1$$

Find the probability generating function (p.g.f.) of X and also find the mean and variance of X using probability generating function (p.g.f.) of X.

(b) Define a stationary process.

Consider the process $X(t) = A_1 + A_2 t$ where A_1, A_2 are independent random variables with $E(A_i) = a_i$, $Var(A_i) = \sigma_i^2$, $i = 1, 2$. Show that the process is not stationary. 2+3=5

(c) Let $\{X_n, n \geq 0\}$ be a Markov chain with three states 0, 1, 2 and with transition matrix

$$\begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

and initial distribution $P\{X_0 = i\} = 1/3, i = 0, 1, 2$

Find $P\{X_2 = 2/X_1 = 1\}$

$$P\{X_2 = 2, X_1 = 1/X_0 = 2\}$$

$$P\{X_2 = 2, X_1 = 1, X_0 = 2\}$$

1+2+2=5

(d) Define periodicity of the states of a Markov chain.

Consider the Markov chain with states 0, 1, 2 having transition matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that the states of the chain are periodic with period 2. 1+4=5

(e) If $\{N(t)\}$ is a Poisson process then prove that the auto-correlation coefficient between $N(t)$ and $N(t+s)$ is $\{t/(t+s)\}^{1/2}$.

(f) The North-Eastern states of India are highly prone to earthquakes. Let us suppose that earthquakes occur at the rate of 2 per year, then

(i) Find the probability that *at least* 3 earthquakes occur during the next two years.

(ii) Find the probability distribution of the time, till the next quake.

2½×2½=5

(g) Write an explanatory note on queueing system.

(h) Obtain the mean number of units in $M/M/1$ queueing model with finite system capacity.

4. Answer ***any three*** of the following questions : 10×3=30

(a) Prove that

(i) The p.g.f. $A(s)$ of the marginal distribution of X is given by $A(s) = P(s, 1)$

(ii) The p.g.f. $B(s)$ of Y is given by $B(s) = P(1, s)$

(iii) The p.g.f. of $(X+Y)$ is given by $P(s, s)$

3+3+4=10

(b) (i) Write a short note on graphical representation of Markov chain. 5

(ii) Consider two brands of tooth paste which are in competition with each other. Let one brand be represented by 0 and the other be represented by 1. Let 'q' be the probability that an individual using a particular brand in the nth year uses the same brand next year, while 'p' is the probability that he changes the brand, where $p+q=1$. Write down the transition probability matrix of the Markov chain. Find what will happen in distant future ? 1+4=5

(c) (i) State and prove the Chapman-Kolmogrov equations. 1+5=6

(ii) Define the following states of Markov chain :
Persistant state, transient state, absorbing state, aperiodic state. 1+1+1+1=4

(d) (i) Prove that, in an irreducible chain, all the states are of the same type. They are either all transient, all persistent null or all persistant non-null. All the states are aperiodic and in the latter case they all have the same period. 5

(ii) Consider a Markov chain having state space $S=\{1,2,3,4\}$ and transition matrix

$$P=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/8 & 1/8 & 1/2 \end{pmatrix}$$

show that all the states of the chain are ergodic. 5

(e) (i) If $\{N(t)\}$ is a Poisson process and $s<t$, then prove that

$$P_r\{N(s)=k/N(t)=n\}=\binom{n}{k}\left(s/t\right)^k\left(1-s/t\right)^{n-k}$$

5

(ii) Prove that the interval between two successive occurrences of Poisson process $\{N(t), t\geq 0\}$ having parameter λ has a negative exponential distribution with mean $\frac{1}{\lambda}$. 5

(f) Under the postulates for Poisson process, prove that $N(t)$ follows Poisson distribution with mean λt i.e. $p_n(t)$ is given by the Poisson law

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots.$$

(g) What do you mean by M/M/1 queueing model with infinite system capacity ? Derive the probability distribution of number of customers in this model. 3+7=10

(h) The arrivals at a counter in a bank occur in accordance with Poisson process at an average rate of 8 per hour. The duration of service of customer has exponential distribution with a mean of 6 minutes. Find the following :

(i) the probability that an arriving customer has to wait,

(ii) the probability that there are three customers in the system,

(iii) the average number of customer in the queue,

(iv) the average waiting time in the queue,

(v) the probability that an arriving customer has to spend less than 15 minutes in the bank.

2+2+2+2+2=10