(d) Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$. 5

7. Answer any *two* parts : 5×2=10

(a) Prove that continuity is a necessary condition for existence of finite derivative of a function.

Show with an example that the condition is not sufficient. 3+2=5

(b) Find the values of a, b, c so that

$$\underset{x \to 0}{\mathrm{Lt}} \frac{a + b\cos x + c\sin x}{x^2}$$

exists and equal to $\frac{1}{2}$. 5

(c) Show that the semivertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$. 5

(d) State the conditions under which a function can be expanded as a Maclaurin's series.

Hence obtain series expansion of

$$f(x) = e^{2x}, \; x \in R$$ 2+3=5

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2016

MATHEMATICS

(Major)

Paper : 4.1

(Real Analysis)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Give an example of a set which is neither an interval nor an open set.

(b) Write true or false :

A finite set has no limit point.

(c) Define a Cauchy sequence.

(d) The positive term series

$$\sum \frac{1}{n^p}$$

is convergent if and only if

(i) $p > 0$

(ii) $p > 1$

(iii) $0 < p < 1$

(iv) $p \le 1$

(Choose the correct answer)

(e) Show that the series

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$$

is not convergent.

(f) If $\sum u_n$ is a series of positive terms, then

(i) convergence of $\sum(-1)^n u_n \Rightarrow$ convergence of $\sum u_n$

(ii) convergence of $\sum u_n \Rightarrow$ convergence of $\sum(-1)^n u_n$

(iii) divergence of $\sum u_n \Rightarrow$ divergence of $\sum(-1)^n u_n$

(iv) convergence of $\sum(-1)^n u_n \Rightarrow$ devergence of $\sum u_n$

(Choose the correct answer)

(g) Let

$$f(x)=\begin{cases}\lambda x^2, & \text{if} \quad x \le 2 \\ 3\ , & \text{if} \quad x > 2\end{cases}$$

If $f(x)$ is continuous at $x=2$, then the value of λ is ______.

(Fill in the blank)

(h) State a condition under which a continuous function will also be uniformly continuous.

(i) If $\underset{x\to\infty}{\text{Lt}}\ f(x)=l$ and $\underset{x\to\infty}{\text{Lt}}\ g(x)$ does not exist, then

(i) $\underset{x\to\infty}{\text{Lt}}\ f(x).g(x)$ does not exist

(ii) $\underset{x\to\infty}{\text{Lt}}\ f(x).g(x)$ exist necessarily

(iii) $\underset{x\to\infty}{\text{Lt}}\ f(x).g(x)$ may or may not exist

(iv) None of the above

(Choose the correct answer)

(j) Write down the geometrical interpretation of Rolle's theorem.

2. Answer the following questions : 2×5=10

(a) Show that the sequence

$$\left\{\frac{2n-7}{3n+2}\right\}$$

is bounded.

(b) Test the convergence of

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

(c) Examine the continuity at $x=1$:

$$f(x)=\begin{cases}2x, & \text{when} \quad 0 \le x < 1 \\ 3\ , & \text{when} \quad x=1 \\ 4x, & \text{when} \quad x > 1\end{cases}$$

(d) In Cauchy's mean value theorem, taking

$$f(x) = \frac{1}{x^2} \text{ and } g(x) = \frac{1}{x} \text{ in } [a, b]$$

show that c is the harmonic mean between a and b.

(e) Examine the differentiability of

$$f(x) = |x| + |x-1| \text{ at } x = 0$$

3. Answer any *four* parts : 5×4=20

(a) Define neighbourhood of a point, limit points of a set and the derived set. Find the limit points of the set

$$\left\{(-1)^n + \frac{1}{n}\right\}$$ 1+1+1+2=5

(b) State Cauchy's first theorem on limit of a sequence. Applying this theorem, prove that

$$\lim_{n \to \infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right] = 0$$ 2+3=5

(c) Test the convergence of the series

$$\frac{2^2}{3^2} + \frac{2^2 . 4^2}{3^2 . 5^2} + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2} + \cdots \infty$$

Apply Gauss's test. 5

(d) Show that the series

$$1 + a + \frac{a(a+1)}{1 . 2} + \frac{a(a+1)(a+2)}{1 . 2 . 3} + \cdots \infty$$

converges for $a \leq 0$ and diverges for $a > 0$. 5

(e) Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

is derivable at $x = 0$ but

$$\operatorname*{Lt}_{x \to 0} f'(x) \neq f'(0)$$ 3+2=5

(f) Expand $\sin x$ in powers of $(x - \pi/2)$ by using Taylor's series. 5

4. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) Prove that the union of an arbitrary family of open sets is open. Does the same result hold for closed sets? Justify your answer. 3+2=5

(b) State Sandwich theorem for sequence of real numbers. Applying this theorem, show that the sequence $\{a_n\}$, where

$$a_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right]$$

converges to zero. 2+3=5

(c) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. 5

(d) Show that the sequence defined by the recursion formula

$$S_{n+1} = \sqrt{3S_n}, \quad S_1 = 1$$

is monotonically increasing and bounded. Is the sequence convergent?

2+2+1=5

5. Answer *either (a)* and *(b) or (c)* and *(d)* : 5×2=10

(a) Define convergence, absolute convergence and conditional convergence of an infinite series. Test the absolute convergence of

$$\sum (-1)^n \frac{n+2}{2^n+5}$$ 1+1+1+2=5

(b) Using comparison test, show that the series

$$\sum(\sqrt{n^4+1} - \sqrt{n^4-1}$$

is convergent. 5

(c) State Cauchy's root test for convergence of an infinite series. Applying this or otherwise test the convergence of

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots \infty$$

1+4=5

(d) Rearranging the terms of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \infty$$

show that the series can be made convergent to different limits.

State a condition under which a series converges to the same limit after rearrangement. 4+1=5

6. Answer any *two* parts : 5×2=10

(a) Prove that if f is a real-valued function defined on a subset of real numbers, then $|f(x)|$ is continuous.

Give an example of a function defined on the set of real numbers which is never continuous, but its absolute value function is always continuous. 3+2=5

(b) If $\underset{x \to a}{\text{Lt}}\ f(x)$ exists, then prove that it must be unique.

Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{4+x} - 2}{x}$. 3+2=5

(c) Prove that a function f defined on an interval I is continuous at $a \in I$ if and only if for every sequence $\{a_n\}$ in I which converges to a, we have

$$\lim_{n \to \infty} f(a_n) = f(a)$$ 5