2016

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following as directed : 1×7=7

(a) Show that in a vector space $V(F)$

$\alpha V = \beta V \Rightarrow \alpha = \beta$

$\alpha V_1 = \alpha V_2 \Rightarrow V_1 = V_2, \quad \alpha, \beta \in F; \quad V, V_1, V_2 \in V$

(b) Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1, 2, 3)$, $\beta = (3, 1, 0)$. Check if the vector $(2, 1, 3)$ is in this subspace.

(c) Examine if the following set S is a subspace of $\mathbb{R}^3$:

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0,\ 2x - y + z = 0\}$$

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined by

$$T(x, y, z) = (x + y + z,\ 2x + y + 2z,\ x + 2y + z)$$

$$(x, y, z) \in \mathbb{R}^3$$

Determine $\ker T$ and its dimension.

(e) For

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

find out the characteristic equation.

(f) If $k \neq 0$ is an eigenvalue of an invertible operator T, show that k^{-1} is an eigenvalue of T^{-1}.

(g) The eigenvalues of a real symmetric matrix

(i) have unit modulus

(ii) are all purely imaginary

(iii) are all real

(iv) are either 0 or 1

(Choose the correct option)

2. Answer the following questions : 2×4=8

(a) Prove that the set of vectors

$$\{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$$

is linearly independent in $\mathbb{R}^3$.

(b) Let V and W be vector spaces over the same field F and $T : V \to W$ be a linear mapping such that $\ker T = \{0\}$. Show that the image of a linearly independent set of vectors in V is also linearly independent in W.

(c) If W is a subspace of a vector space V, then show that $L(W) = W$ and conversely. ($L(W)$: linear span of W).

(d) Show that two eigenvectors of a square matrix A over a field F corresponding to two distinct eigenvalues of A are linearly independent.

3. Answer any *one* part : 5

(a) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ and $\{(1, 0), (1, 1)\}$ of $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$

Find T.

(b) Find range, rank, ker and nullity of the linear transformation defined by

$$T:\mathbb{R}^3 \to \mathbb{R}^2$$

such that

$$T(x, y, z) = (x+y,\ 2z-x),\ (x,\ y,\ z) \in \mathbb{R}^3$$

4. Answer the following questions : 10×2=20

(a) Prove that a finite set of non-zero vectors $\{v_1, v_2, \cdots, v_n\}$ in a vector space $V(F)$ is linearly dependent if and only if there exists v_k, $2 \le k \le n$, such that v_k is a linear combination of $v_1, v_2, \cdots, v_{k-1}$.

Or

If W is a subspace of a finite dimensional vector space V, prove that

$$\dim \frac{V}{W} = \dim V - \dim W$$

(b) Let V be the space of $n \times n$ matrices over F. Let A be a fixed $n \times n$ matrix over F, and T is a linear operator 'left multiplication by A' on V. Prove that both A and T have the same eigenvalues.

Or

Let a, b, c be elements of a field F and

$$A = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}$$

Prove that the characteristic polynomial of A is same as that of its minimal polynomial.

GROUP—B

(Vector)

(*Marks* : 40)

5. Answer the following : 1×3=3

(a) Find the volume of the parallelopiped whose coterminous edges are represented by

$$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k},\ \vec{b} = \hat{i} + 2\hat{j} - \hat{k},\ \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$$

(b) Show that the vectors $\vec{a} \times (\vec{b} \times \vec{c})$, $\vec{b} \times (\vec{c} \times \vec{a})$, $\vec{c} \times (\vec{a} \times \vec{b})$ are coplanar.

(c) If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors such that $\vec{a} = \vec{c} + \vec{d}$, where $\vec{c}$ is a vector parallel to $\vec{b}$, and $\vec{d}$ is a vector perpendicular to $\vec{b}$, obtain the expressions for $\vec{c}$ and $\vec{d}$ in terms of $\vec{a}$ and $\vec{b}$.

6. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three unit vectors such that

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$$

find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, $\vec{b}$ and $\vec{c}$ being non-parallel. 2

7. Answer the following questions : 5×3=15

(a) If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, then prove that $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$, $\vec{a} \times \vec{b}$ are also non-coplanar. Also express $\vec{a}$, $\vec{b}$, $\vec{c}$ in terms of $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$, $\vec{a} \times \vec{b}$.

Or

Prove that

$$|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2a^2$$

where $a = |\vec{a}|$.

(b) If $\vec{u} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{v} = \sin t\hat{i} - \cos t\hat{j}$, evaluate

(i) $\frac{d}{dt}(\vec{u} \cdot \vec{v})$

(ii) $\frac{d}{dt}(\vec{u} \cdot \vec{u})$

(iii) $\frac{d}{dt}(\vec{u} \times \vec{v})$

(c) Taking $\vec{F} = x^2y\hat{i} + xz\hat{j} + 2yz\hat{k}$, verify that div curl $\vec{F} = 0$.

8. Answer the following questions : 10×2=20

(a) Prove that $\text{curl}[\vec{r} \times (\vec{a} \times \vec{r})] = 3\vec{r} \times \vec{a}$, where $\vec{a}$ is a constant vector and

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Or

Prove that the necessary and sufficient condition for a vector $\vec{v}(t)$ to have a constant direction is

$$\vec{v} \times \frac{d\vec{v}}{dt} = \vec{0}$$

(b) Evaluate

$$\int \vec{F} \cdot d\vec{r}$$

along the curve $x^2 + y^2 = 1$, $z = 1$ in the positive direction from (0, 1, 1) to (1, 0, 1), where $\vec{F} = (2x + yz)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}$.

Or

Evaluate

$$\iint_S \vec{F} \cdot \hat{n}\, dS$$

where $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and S is that portion of the plane $x + y + z = 1$ which lies in the first octant.

★★★